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Contributions to Linear System Theory

by

Leonard Weiss[‡] and R.E. Kalman[‡‡]

1. Introduction.

The characterization of a physical system, i.e., the cataloging of the significant facts concerning its physical behavior generally suggests the mode of representation of the system for purposes of analysis.

Newtonian mechanics as well as quantum mechanics is based on the concept of state for describing physical systems. In the special case where the state space is finite-dimensional and the evolution of the system is governed by a differential or difference equation, a large body of theory has been developed over the past few years ("state-variable" techniques), especially with respect to problems involving control systems [1, 2]. Despite the increasing emphasis on the "state-variable" approach, much of the literature on linear systems continues to be written in terms of input-output relations, since, in many situations, the latter is a natural mode of system description. The parallel developments of the input-output and the state-variable approaches to system analysis have left much in their wake that is either not well understood or imprecisely explained.

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The purpose of this paper is to give a self-contained exposition of some important facets of linear system theory which goes beyond what is presently available in the literature. Specifically, our concern is with the class of dynamical systems which are governed by a set of linear ordinary differential equations of finite order in the state variables. New results (see [3]) are presented in the theory of controllability, observability, and canonical decomposition of such systems. These results are used in a detailed study of the relation of the state-variable representation of a system to certain well known types of time-domain input-output representations.

2. The Primary Representation.

2.1. Definition of a system.

For completeness, we give an axiomatic definition of a system to which all representations can be related. The system of axioms we shall adopt is a modified version of that previously stated in [1]. The purpose of these axioms is to motivate to some extent the concept of the state. We shall not be concerned, however, with a technical analysis of these axioms, and will restrict all future considerations to the analysis of the "state-variable" differential equation (2.1). In short, the axioms now to be stated will serve as motivation for taking the differential equation (2.1) as the primary representation of a system.

DEFINITION 1. By a dynamical system we shall mean a mathematical structure described by the following axioms:

(i) There is given an abstract space, Σ, called the state space and a set $\mathcal{T}$ of values of time at which the behavior of the system is defined. $\mathcal{T}$ is an ordered subset of the real numbers, with the usual ordering $>$ [or $<$]. If $t_1, t_0 \in \mathcal{T}$, the statement $t_1 > t_0$ [or $t_1 < t_0$] will mean that t_1 is in the future [or in the past] with respect to t_o; equivalently, t_o is in the <u>past</u> [or in the <u>future</u>] with respect to t_1.

(ii) There is given an abstract space Ω of functions of time $u: \mathcal{T} \to \mathcal{U}$ (where $\mathcal{U}$ is a Euclidean space) which represent the admissible <u>inputs</u> to the system.

(iii) For any initial time $t \in \mathcal{T}$, any initial state $x \in \Sigma$, and any input $u \in \Omega$ defined for $t \geqq \tau$ [or $t \leqq \tau$], states at other values of time of the system are determined by a given <u>transition function</u> $\varphi: \Omega \times \mathcal{T} \times \mathcal{T} \times \Sigma \to \Sigma$, which is written as $\varphi_u(t; \tau, x)$. This function has the following properties:

(a) $\varphi_u(\tau; \tau, x) = x$ <u>for any</u> $u \in \Omega$, $\tau \in \mathcal{T}$, $x \in \Sigma$.

(b) $\varphi_u(t; \tau, x)$ <u>is defined only when</u> $t \geqq \tau$ [or $t \leqq \tau$].

(c) $\varphi_u(t_2; t_o, x) = \varphi_u(t_2; t_1, \varphi_u(t_1; t_o, x))$ <u>for all</u> $u \in \Omega$, <u>all</u> t_o, t_1, t_2 <u>in</u> $\mathcal{T}$ <u>such that</u> $t_2 \geqq t_1 \geqq t_o$ [<u>or</u> $t_2 \leqq t_1 \leqq t_o$], <u>and all</u> $x \in \Sigma$.

(d) <u>If</u> $u_{[\tau, t]}$ <u>denotes the equivalence class of functions</u> $v \in \Omega$ <u>whose values agree with</u> u <u>on the set</u> $[\tau, t] \cap \mathcal{T}$, <u>then</u>

$$\varphi_u(t; \tau, x) = \varphi_{u_{[\tau, t]}}(t; \tau, x).$$

(iv) Every output of the system at time t is given by the value of a real function $\psi: \mathcal{T} \times \Sigma \to R$; where ψ belongs to a given class Y.†

(v) The functions φ and ψ are continuous with respect to suitable topologies defined on Σ, $\mathcal{T}$, Ω, and the reals, as well as the induced product topologies.

---

† A more general approach would be to make the output depend on the input as well as the state. The corresponding extension of all results in this paper to hold for this case is straightforward.

It is understood that the preceding axioms define a mathematical structure, not a physical system. It is found, however, that the behavior of complex interconnected objects obeying the laws of classical physics can usually be described in terms of these axioms. By a "system" we shall always mean a dynamical system in the sense of Definition 1. The modification of the axiom system in Definition 1 from that in reference [1] consists in allowing explicit consideration of the two opposite orderings of the set $\mathcal{T}$. The repercussions of this seemingly minor point turns out to be of significance, as will be seen shortly.

From now on we shall restrict attention to a special class of systems which are:

(i) finite-dimensional (Σ = finite-dimensional)

(ii) continuous-time ($\mathcal{T}$ = real line and φ, ψ = smooth real functions of t)

(iii) linear (ψ is linear in x and φ is linear jointly in x and u)

(iv) multi-input, multi-output ($\mathcal{U}$ is p dimensional, Y has r elements).

Under these special assumptions, it can be proved [4] that the transition function of the most general dynamical system which satisfies the above axioms is a solution of the vector differential equation:

$$(2.1) \qquad \begin{aligned} dx/dt &= F(t)x + G(t)u(t) \\ y(t) &= H(t)x(t) \end{aligned}$$

where

$$
\begin{aligned}
x &= \text{state vector (real n-vector)} \\
y(t) &= \text{output}^{\dagger} \quad \text{(real r-vector)} \\
u(t) &= \text{input}^{\dagger} \quad \text{(real p-vector)} \\
F(t) &= n \times n \text{ real matrix} \\
G(t) &= \text{real } \; n \times p \text{ matrix} \\
H(t) &= \text{real } \; r \times n \text{ matrix}
\end{aligned}
$$

and, for convenience, $F(\cdot)$, $G(\cdot)$, $H(\cdot)$, are C^{∞} functions of t.*

We shall call (2.1) the <u>state-variable</u> differential equation.

2.2. <u>Solution of the State-Variable Differential Equation.</u>

Given the initial condition $x(t_o) = x_o$, it is well known [5] that the solution of (2.1) is uniquely defined and is given by

$$
x(t) = X(t)X^{-1}(t_o)x_o + \int_{t_o}^{t} X(t)X^{-1}(\tau)G(\tau)u(\tau)d\tau, \tag{2.2}
$$

where $X(\cdot)$ is a fundamental matrix solution [5] of the homogeneous equation

$$
dx/dt = F(t)x, \tag{2.3}
$$

i.e.,

$$
\frac{dX}{dt} = F(t)X, \quad \det X(t_o) \neq 0.
$$

*For many arguments C^n or even C^o is sufficient

† From now on, unless otherwise specified, the word "input" denotes "vector input" and the word "output" denotes "vector output".

To place (2.2) in direct correspondence with our axioms, we write (2.2) in terms of the transition matrix $\Phi(t, \tau)$, which is defined by

$$\Phi(t, \tau) = X(t)[X(\tau)]^{-1}. \tag{2.4}$$

This definition is equivalent to stating that Φ satisfies the relations

$$d\Phi/dt = F(t)\Phi, \quad \Phi(t_o, t_o) = I.$$

Then (2.2) becomes

$$x(t) = \Phi(t, t_o)x_o + \int_{t_o}^{t} \Phi(t, \tau)G(\tau)u(\tau)d\tau, \tag{2.5}$$

and $y(t)$ is then given explicitly as

$$y(t) = H(t)\Phi(t, t_o)x_o + \int_{t_o}^{t} H(t)\Phi(t, \tau)G(\tau)u(\tau)d\tau. \tag{2.6}$$

It is easy to verify that the function φ defined by

$$\varphi_u(t; t_o, x_o) = x(t) \tag{2.7}$$

where $x(t)$ is given by (2.5), satisfies axiom (iii). The remaining axioms are also easy to verify.

It is important to remark that the function φ defined above not only satisfies axiom (iii) but has the additional property of being defined for <u>all</u> t, t_o, not merely $t \geqq t_o$ or $t \leqq t_o$. Since φ can be regarded as a properly defined transition function with either ordering in time, it is clear that specification of the ordering is an essential point in interpreting a differential equation as a dynamical system.

2.3. C-Systems and A-Systems.

If the time is ordered in the usual physical way, ($>$ in Def. 1) then axiom (iii-d) follows from what is usually called the Principle of Causality: the future has no effect on the present or past. More precisely, we shall say

DEFINITION 2. A causal (C-) system is characterized by the property that the behavior of the system at any time t_o is completely determined by events in the past and present, i.e., at $t \leq t_o$.

DEFINITION 3. An anticausal (A-) system is characterized by the property that the behavior of the system at time t_o is completely determined by events in the present and future, i.e., at $t \geq t_o$.

In view of the axioms given above, we may say that

(i) In a C-system the state at time t_o summarizes the present and the past history of the system.

(ii) In an A-system, the state at time t_o summarizes the present and the future history of the system.

Whether a given system is of type C or A depends solely on the ordering of the set $\mathcal{T}$. As was seen above, by merely writing down the differential equations of the system no choice of the time ordering is implied; this ordering must always be specified separately.* It is not even obvious at present whether or not there is a natural direction of the flow of time in the physical world [6].

Fig. 1 gives a graphical illustration of the interpretation of a differential equation (2.1) as a C or A system.

---

* In writing down formula (2.5), it is often tacitly assumed that $t > t_o$. This means a particular ordering has been chosen. But the formula is invariant with respect to interchanging t and t_o, which shows that the adopted ordering is arbitrary and not a property of the differential equation.

One might conceive of a system which is noncausal, i.e., neither of type C nor A. An example which is frequently quoted is that of the ideal low pass filter [2] whose behavior at any given time depends on both past and future stimuli. An ideal low-pass filter is not a dynamical system in the sense of Definition 1. It is not at all illogical for some "frequency-response" functions to not define a dynamical system. After all, some characteristic functions (namely those whose inverse Fourier transform has negative values) do not define probability density functions, and therefore such "characteristic" functions have nothing to do with random variables!

The signficance of the role played by noncausal systems in system theory is not presently well established. For this reason, we make no further references to such systems in this paper.

2.4. Controllability and Observability.

The concepts mentioned in the title have come to play an important role in the development of system theory, especially in the theory of control systems (see reference [1]). Our purpose here is to define these concepts and show how they relate to system structure.

It is interesting to note that there are two natural ways of looking at each concept [3] although the past literature has been mainly concerned with only one of these ways. The choice made between them will, in general, depend on the type of information sought. The definitions are as follows.

Assume a given ordering of the set $\mathcal{T}'$. Then

DEFINITION 4. Consider a system (2.1).

(i) The couple (x_o, t_o) (hereafter referred to as a "phase" of the system) is said to be causal (C-) controllable [or anticausal (A-) controllable] if there exists some finite $t_1 > t_o$ [or $t_{-1} < t_o$] and some input $u_{[t_o, t_1]}$ [or $u_{[t_{-1}, t_o]}$] which transfers (x_o, t_o) to $(0, t_1)$ [or (x_o, t_o) to $(0, t_{-1})$].

(ii) A phase which is both C-controllable and A-controllable is <u>bidirectionally</u> (<u>B</u>-) <u>controllable</u>.

(iii) If every phase (x, t_o) is C- [or A- or B-] controllable $\forall x \in \Sigma$, then the system is <u>C</u>- [or <u>A</u>- or <u>B</u>-] <u>controllable at</u> $\underline{t_o}$.

(iv) If every phase is C- [or A- or B-] controllable, the system is <u>completely</u> <u>C</u>- [or <u>A</u>- or <u>B</u>-] <u>controllable</u>.

(v) If the control intervals in (i) above can be made arbitrarily small, we speak of <u>differential controllability</u> over those intervals.*

DEFINITION 5. Consider a system (2.1). Assume $u(t) \equiv 0$.

(i) The system is <u>causal</u> (<u>C</u>-) <u>observable</u> [or <u>anticausal</u> (<u>A</u>-) <u>observable</u>] <u>at time</u> $\underline{t_o}$ if the state of the system at time t_o can be identified from knowledge of the system's output over a finite interval $[t, t_o]$ (or $[t_o, t]$).

(ii) A system which is both C-observable and A-observable at t_o is <u>bidirectionally</u> [<u>B</u>-] <u>observable at</u> $\underline{t_o}$.

(iii) If the system is C- [or A- or B-] observable at every $t \in \mathcal{T}$, it is <u>completely</u> <u>C</u>- [or <u>A</u>- or <u>B</u>-] <u>observable</u>.

(iv) If the intervals of output observation mentioned in (i) can be made arbitrarily small, we speak of <u>differential observability</u> over those intervals.

Notes on the definition of observability:

1. Intuitively, there appears to be a connection between differential observability at a point t_o and the ability to identify a state from a finite number of derivatives of the output at t_o. In a later section, this intuitive connection is made explicit.

---

*Differential controllability is a concept introduced by LaSalle [7], who called it "complete controllability". Since then, it has become more common to use "complete controllability" in the manner employed here.

2. The motivation for the appelatives of "causal" and "anticausal" in the definition of observability is as follows. In a system which is A-observable at the "present" instant, one cannot identify the "present" state without knowledge of future inputs. It may seem that there exist identification procedures for "physical" systems which are based on our "anticausal" observability property [8]. However, in such cases, the principle of causality imposes a built-in delay into the procedure so that in fact it is not the "present" state which can be identified, but only a "past" state.

2.5. Controllability and Observability of the System (2.1).

In this section, we give necessary and sufficient conditions for the system (2.1) to possess certain controllability and observability properties. We begin with

THEOREM 1. The system (2.1) is C-controllable at t_o if and only if there exists $t_1 > t_o$ such that the rows of the matrix $\Phi(t_o, \cdot)G(\cdot)$ are linearly independent functions over the interval $[t_o, t_1]$.

Proof. (Sufficiency): If the rows of $\Phi(t_o, \cdot)G(\cdot)$ are linearly independent functions on $[t_o, t_1]$, the matrix

$$C(t_o, t_1) = \int_{t_o}^{t_1} \Phi(t_o, t)G(t)G'(t)\Phi'(t_o, t)dt$$

where the "prime" indicates transpose, is positive definite. Then, setting

$$u(t) = - G'(t)\Phi'(t_o, t)C^{-1}(t_o, t_1)x_o$$

in (2.5), we obtain $x(t_1) = 0$.

(Necessity): If there is no $t_1 > t_o$ such that the rows of $\Phi(t_o, \cdot)G(\cdot)$ are linearly independent functions on $[t_o, t_1]$, then there is a vector x_1 in the state space such that

$$x_1'\Phi(t_o, t)G(t) \equiv 0 \text{ for all } t > t_o.$$

From (2.5),

$$\begin{aligned} x_1'\Phi(t_o, t)x(t) &= x_1'x_o + x_1' \int_{t_o}^{t} \Phi(t_o, \tau)G(\tau)u(\tau)d\tau \\ &= x_1'x_o. \end{aligned}$$

Clearly, the component of x in the direction of x_1 is uncontrollable for $t > t_o$ since the term involving u is zero. But the system was assumed to be C-controllable at t_o. Hence we have a contradiction. Q.E.D.

THEOREM 2. <u>The system</u> (2.1) <u>is</u> <u>A</u>-<u>controllable at</u> t_o <u>if and only if there exists</u> $t_{-1} < t_o$ <u>such that the rows of the matrix</u> $\Phi(t_o, \cdot)G(\cdot)$ <u>are linearly independent functions on the interval</u> $[t_{-1}, t_o]$.

<u>Proof</u>. Same as for Theorem 1 with t_1 replaced by t_{-1} and ">" replaced by "<".

THEOREM 3. <u>The system</u> (2.1) <u>is</u> <u>C</u>-<u>observable at</u> t_o <u>if and only if there exists</u> $t_{-1} < t_o$ <u>such that the columns of the matrix</u> $H(\cdot)\Phi(\cdot, t_o)$ <u>are linearly independent functions over the interval</u> $[t_{-1}, t_o]$.

<u>Proof</u>. (Sufficiency): Consider (2.1) with $u(\cdot) = 0$. Then, from (2.1) and (2.5),

$$y(t) = H(t)x(t) = H(t)\Phi(t, t_o)x_o. \tag{2.8}$$

Multiplying both sides by $\Phi'(t, t_o)H'(t)$ and integrating over $[t_{-1}, t_o]$ yields

$$\int_{t_{-1}}^{t_o} \Phi'(t, t_o)H'(t)y(t)dt = \int_{t_{-1}}^{t_o} \Phi'(t, t_o)H'(t)H(t)\Phi(t, t_o)dt \cdot x_o$$

$$= D(t_o, t_{-1})x_o$$

where

$$D(t_o, t_{-1}) = \int_{t_{-1}}^{t_o} H(t)\Phi(t, t_o)[H(t)\Phi(t, t_o)]'dt. \tag{2.9}$$

Clearly, x_o is identifiable from knowledge of $y(\cdot)$ over $[t_{-1}, t_o]$ if D is nonsingular. The nonsingularity of D follows from (2.9) if the columns of $H(\cdot)\Phi(\cdot, t_o)$ are linearly independent functions on $[t_{-1}, t_o]$.

(Necessity): If there is no $t_{-1} < t_o$ such that the columns of $H(\cdot)\Phi(\cdot, t_o)$ are linearly independent functions on $[t_{-1}, t_o]$, then from (2.8), there exists x_o such that $y(t) \equiv 0$ for $t \leq t_o$. The system is then not C-observable at t_o, and we have a contradiction. Q.E.D.

THEOREM 4. <u>The system</u> (2.1) <u>is A-observable at</u> t_o <u>if and only if there exists</u> $t_1 > t_o$ <u>such that the columns of the matrix</u> $H(\cdot)\Phi(\cdot, t_o)$ <u>are linearly independent functions on</u> $[t_o, t_1]$.

<u>Proof</u>. Same as for Theorem 2 with t_{-1} replaced by t_1 and "$<$" replaced by "$>$".

As immediate consequences of Theorems 1, 2 and 3, 4 respectively, we have

COROLLARY 1.* A system (2.1) is differentially controllable on an interval I if and only if the rows of the matrix $\Phi(t_o, \cdot)G(\cdot)$ are linearly independent functions on every subinterval of I.

COROLLARY 2. A system (2.1) is differentially observable on an interval I if and only if the columns of the matrix $H(\cdot)\Phi(\cdot, t_o)$ are linearly independent functions on every subinterval of I.

In turn, Corollaries 1, 2 give us (see [3])

THEOREM 5. A system (2.1) is differentially controllable on an interval I if and only if the rows of the matrix $\Phi(t_o, \cdot)G(\cdot)$ are linearly independent functions over a subinterval of I and, in addition, there exists a vector linear differential equation, defined on I and having no singularities on I, for which the rows of the matrix $\Phi(t_o, \cdot)G(\cdot)$ are solutions.

THEOREM 6. A system (2.1) with $u(t) \equiv 0$ is differentially observable on an interval I if and only if the columns of the matrix $H(\cdot)\Phi(\cdot, t_o)$ are linearly independent over a subinterval of I and in addition there exists a vector linear differential equation, defined on I and having no singularities on I, for which the columns of the matrix $H(\cdot)\Phi(\cdot, t_o)$ are solutions.

Proof of Theorem 5. (Sufficiency): Let $G'(\cdot)\Phi'(t_o, \cdot)$ be the transpose of $\Phi(t_o, \cdot)G(\cdot)$. Suppose the columns of $\{R_i\}$ of $G'(\cdot)\Phi'(t_o, \cdot)$ are solutions of the differential equation

$$\sum_{i=1}^{n} A_i(t)v^{(i)} = 0 \tag{2.11}$$

* This result was originally proved by LaSalle [7]. In his terminology, a differentially controllable system is a "proper" system.

defined on I, where v is a p-vector, $v^{(i)} = \frac{d^i v}{dt^i}$, and $A_i(\cdot)$ is a $p \times p$ matrix of suitably differentiable time functions with $\det A_n(t) \neq 0 \ \forall t \in I$. There will be $n \cdot p$ linearly independent vector solutions of (2.11) $v_1, \ldots, v_{n \cdot p}$. It is easy to show that (2.11) can also be written as a set of $n \cdot p$ first-order vector equations whose solutions correspond to the $\{v_i\}$. A fundamental matrix for the latter set of equations is given by

$$\begin{bmatrix} v_1 & \cdots & v_n & v_{n+1} & \cdots & v_{n \cdot p} \\ v_1^{(1)} & \cdots & v_n^{(1)} & v_{n+1}^{(1)} & \cdots & v_{n \cdot p}^{(1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ v_1^{(p-1)} & \cdots & v_n^{(p-1)} & v_{n+1}^{(p-1)} & \cdots & v_{n \cdot p}^{(p-1)} \end{bmatrix}. \tag{2.12}$$

Now, a subset of the $\{v_i\}$ spans the linear manifold defined by the $\{R_i\}$. Suppose we replace $v_1, \ldots, v_n$ in (2.12) by $R_1, \ldots, R_n$ respectively, and examine the determinant associated with the resulting solution matrix. By a well-known theorem [5] if the determinant of a solution matrix of a set of linear equations defined on an interval I vanishes anywhere on I, it must vanish everywhere on I. Conversely, if the determinant is nonzero at one point in I, it must be nonzero for all points in I. The former case implies that two columns of the matrix (2.12) are linearly dependent over I. But the $\{v_i\}$ were presumed linearly independent, and by hypothesis, the $\{R_i\}$ are linearly independent over the interval I. Therefore, the determinant associated with (2.12) with $R_1, \ldots, R_n$ replacing $v_1, \ldots, v_n$ is nonzero for all $t \in I$ which implies that the $\{R_i\}$ are linearly independent functions over every subinterval of I. Application of Corollary 1 completes this part of the proof.

(Necessity): If the system (2.1) is differentially controllable on I, then by Corollary 1 the $\{R_i\}$ must be linearly independent on every subinterval of I. By an easy extension of the argument for the scalar case, one has the following result from the theory of differential equations [5]. Given a set of p-vectors $v_1(\cdot), \ldots, v_{n\cdot p}(\cdot)$ which are linearly independent functions on every subinterval of a given interval I, so that the determinant associated with (2.12) vanishes nowhere on I. Then the $\{v_i\}$ satisfy a linear homogeneous differential equation, defined on I and with no singularities on I, which is given implicitly by

$$\det \begin{bmatrix} v_1 & \cdots & v_{n\cdot p} & v \\ v_1^{(1)} & \cdots & v_{n\cdot p}^{(1)} & v^{(1)} \\ \vdots & & \vdots & \vdots \\ v_1^{(p)} & \cdots & v_{n\cdot p}^{(p)} & v^{(p)} \end{bmatrix} = 0 \tag{2.13}$$

where v is the dependent variable (a p-vector). We now merely associate the $\{R_i\}$ with n members of the $\{v_i\}$ and the theorem is proved. Q.E.D.

Theorem 6 is proved by strict analogy.

2.6. The Canonical Structure of a Linear Dynamical System.

It is known [9], [10], that one can use the concepts of (C-) controllability and (C-) observability to form a direct sum decomposition of the state space of a system (2.1). The fact that two types of controllability and observability can be defined leads to an immediate extension of previous results [3]. We now give

THEOREM 7. (i) Consider a linear dynamical system (2.1). For a given ordering of the set $\mathcal{T}$, and at every fixed instant t of time, there is a coordinate system in the state space relative to which the components of the state vector can be decomposed in any one of four ways into a direct sum of four parts

$$x = x^{a} \oplus x^{b} \oplus x^{c} \oplus x^{d},$$

which correspond respectively to the schemes I through IV below.

$$\text{I}\quad \left\{\begin{array}{ll} \text{Part (a):} & \text{C-controllable but C-unobservable} \\ \text{Part (b):} & \text{C-controllable and C-observable} \\ \text{Part (c):} & \text{C-uncontrollable and C-unobservable} \\ \text{Part (d):} & \text{C-uncontrollable but C-observable,} \end{array}\right.$$

$$\text{II}\quad \left\{\begin{array}{ll} \text{Part (a):} & \text{A-controllable but A-unobservable} \\ \text{Part (b):} & \text{A-controllable and A-observable} \\ \text{Part (c):} & \text{A-uncontrollable and A-unobservable} \\ \text{Part (d):} & \text{A-uncontrollable but A-observable.} \end{array}\right.$$

$$\text{III}\quad \left\{\begin{array}{ll} \text{Part (a):} & \text{C-controllable but A-unobservable.} \\ \text{Part (b):} & \text{A-controllable and C-observable} \\ \text{Part (c):} & \text{A-uncontrollable and C-unobservable} \\ \text{Part (d):} & \text{A-uncontrollable but C-observable.} \end{array}\right.$$

$$\text{IV}\quad \left\{\begin{array}{ll} \text{Part (a):} & \text{A-controllable but C-unobservable} \\ \text{Part (b):} & \text{A-controllable and C-observable} \\ \text{Part (c):} & \text{A-uncontrollable and C-unobservable} \\ \text{Part (d):} & \text{A-uncontrollable but C-observable.} \end{array}\right.$$

(ii) Relative to such a choice of coordinates, and for any type of decomposition the system matrices have the canonical form

$$(2.14)\qquad F(t) = \begin{bmatrix} F^{aa}(t) & F^{ab}(t) & F^{ac}(t) & F^{ad}(t) \\ 0 & F^{bb}(t) & 0 & F^{bd}(t) \\ 0 & 0 & F^{cc}(t) & F^{cd}(t) \\ 0 & 0 & 0 & F^{dd}(t) \end{bmatrix}$$

$$G(t) = \mathrm{col}(G^{a}(t),\ G^{b}(t),\ 0,\ 0)$$
$$H(t) = (0,\ H^{b}(t),\ 0,\ H^{d}(t)).$$

The development of the statements contained in Theorem 7 follow in parallel fashion the development given in [1] in which only the type I decomposition was discussed. Figure 2 gives a graphical picture of the canonical structure for a type I, II, III or IV decomposition

Let $n_a(t)$, $n_b(t)$, $n_c(t)$, $n_d(t)$ be the dimension numbers for parts a, b, c, d respectively of a given type of decomposition for a system (2.1). The dimension, n, of the state space of (2.1) is then given by

$$(2.15)\qquad n = n_a(t) + n_b(t) + n_c(t) + n_d(t) \quad \forall t.$$

Although (2.15) holds for any type of decomposition, in the general time-varying case the dimension numbers (at any given value of t) for one type of decomposition need not coincide with those of another type. It can be shown, however, that the dimension numbers for all of the above decompositions are constants if (2.1) is periodic or analytic.

To illustrate the significance of a dimension number being constant, suppose $n_a(t)$ is constant for a given type of decomposition of (2.1). Then part (a) of the associated decomposition is a subsystem of (2.1). If the decomposition is of type II, then this subsystem is A-controllable but A-unobservable for all t.

In a later section, we shall investigate the conditions under which the "b" part of a given decomposition has constant dimension. This problem is connected with the study of the relationship between state-variable and input-output system representations.

3. System Representation by Input-output Relations in the Time-Domain - The Weighting Pattern.

In this section we explore one type of input-output representation of linear systems. Specifically, we shall relate certain properties of the latter to properties normally associated with the internal structure of a system.

3.1. Definition of Weighting Pattern.

DEFINITION 6. The weighting pattern[†] $W(t, \tau)$ for the dynamical system (2.1) is defined by the relation

$$y(t) - H(t)\Phi(t, t_o)x_o = \int_{t_o}^{t} W(t, \tau)u(\tau)d\tau. \tag{3.1}$$

That is, knowing the state of a C-system [or A-system] at any value t_o of time, as well as the input over an interval $[t_o, t_1]$ [or the interval $[t_{-1}, t_o]$] enables one to establish by means of the $(r \times p$ matrix) weighting pattern, the output of the system at time t_1 [or t_{-1}]. From (3.1), (2.6), and (2.4) we have, in view of the uniqueness of solutions of the differential equation (2.1),

[†] This terminology is due to Prof. W. H. Huggins. Unless otherwise specified, the term "weighting pattern" implies "weighting pattern matrix".

$$W(t, \tau) = H(t)\Phi(t, \tau)G(\tau) = H(t)X(t)[X(\tau)]^{-1}G(\tau) \quad \forall t, \tau. \tag{3.2}$$

If the initial state is $\underline{0}$ (3.1) becomes

$$y(t) = \int_{t_o}^{t} W(t, \tau)u(\tau)d\tau. \tag{3.3}$$

Hence, for zero initial conditions, the weighting pattern contains all the information needed to completely describe the input-output relation of the system (2.1).

Since (2.1) can represent either a C-system of an A-system, it is clear that the weighting pattern must also be independent of the ordering of $\mathcal{T}$.

As is well known, one can formally identify the i^{th} column of $W(t, \tau)$ (i.e., $W_i(t, \tau)$) as the "output" of the system (2.1) at time t corresponding to an "input" all of whose components are zero except the i^{th} which is a unit impulse applied at time τ (assuming that at time τ the state of the system is the origin). The formal derivation of this result is as follows. In (2.6) let $x_o = 0$ and

$$u_\tau(\xi) = \begin{bmatrix} 0 \\ \vdots \\ \delta(\xi - \tau) \\ 0 \\ \vdots \\ 0 \end{bmatrix} (i^{th}), \quad \tau \in [t_o, t].$$

Then

$$y(t, \tau) = \int_{t_0}^{t} H(t)\Phi(t, \xi)G(\xi)u_\tau(\xi)d\xi \tag{3.4}$$

$$= \int_{t_0}^{t} H(t)\Phi(t, \xi)G_i(\xi)\delta(\xi - \tau)d\xi$$

where G_i is the i^{th} column of G. By the well known "sifting" property of the δ-function,

$$y(t, \tau) = H(t)\Phi(t, \tau)G_i(\tau),$$

$$= W_i(t, \tau).$$

The weighting pattern (matrix) is therefore often referred to as the impulse response (matrix) of the system.†

3.2. Causal and Anticausal Impulse Responses.

A causal system can be described by stating that there is no response prior to an excitation. This leads to the requirement that the impulse response function vanish for $\tau > t$. In like manner the impulse response function for an anticausal system must vanish for $\tau < t$. We now give the following definitions.

DEFINITION 7. The causal impulse response (C-impulse response) of the dynamical system (2.1) is given by

$$\begin{aligned} W_C(t, \tau) &= W(t, \tau) \quad \text{if} \quad t \geq \tau, \\ &= 0 \qquad\qquad \text{if} \quad t < \tau. \end{aligned} \tag{3.5}$$

† From now on, the word "matrix" will be understood when "impulse responses" are discussed.

DEFINITION 8. The <u>anticausal</u> (<u>A</u>-) <u>impulse response</u> of the dynamical system (2.1) is given by

$$(3.6) \qquad W_A(t, \tau) = \begin{cases} 0 \quad , & \text{if } \ t > \tau. \\ -W(t, \tau), & \text{if } \ t \leq \tau. \end{cases}$$

In view of definition 7, for a causal system we can always write (3.1) in the form

$$(3.7) \qquad y(t) = H(t)\Phi(t, t_o)x_o + \int_{t_o}^{\infty} W_C(t, \tau)u(\tau)d\tau.$$

For certain purposes (see [11]), it is convenient to write the right hand side of (3.7) as a single integral

$$(3.8) \qquad y(t) = \int_{-\infty}^{\infty} W_C(t, \tau)u(\tau)d\tau.$$

In order to do this, we must have

$$(3.9) \qquad H(t)\Phi(t, t_o)x_o = \int_{-\infty}^{t_o} W_C(t, \tau)u(\tau)d\tau$$

for a suitable definition of the function $u(\cdot)$ on $(-\infty, t_o)$. It follows that we can always do this as long as the dimension number $n_d(t)$ for a type III canonical decomposition is zero for all t. (See Theorem 21.) Similarly, for an anticausal system, we can always write

$$y(t) = \int_{-\infty}^{\infty} W_A(t, \tau)u(\tau)d\tau$$

as long as the dimension number $n_d(t)$ for a type II canonical decomposition is zero for all t.

3.3. Weighting Patterns and Realizations.

Suppose we are given a matrix function of two variables t, τ and we regard this function as a weighting pattern W, or impulse response W_C or W_A. In order to relate such an abstractly given weighting pattern to a dynamical system, we introduce

DEFINITION 9. A dynamical system (2.1) is a local realization of W [or W_C or W_A] on an interval I whenever (3.2) [or (3.2 and 3.5) or (3.2) and 3.6)] holds on I. If the aforementioned relation holds for all I, the realization is global.

The term "realization" is motivated by the fact that it is possible to build real (physical) systems (analog computers, etc.) which obey (2.1) to any desired degree of accuracy.

We now give the following result which is stated for global realizations but which has an obvious counterpart for local realizations.

THEOREM 8. A function of two variables, W, is a weighting pattern realizable by means of a dynamical system (2.1) if and only if there exist two matrix functions of time $\Psi(\cdot)$ and $\Theta(\cdot)$ defined over $-\infty < t < \infty$ such that for all t, τ we have

$$W(t, \tau) = \Psi(t)\Theta(\tau). \tag{3.10}$$

Proof. If condition (3.10) holds, then the system defined by

$$F(t) \equiv 0, \quad G(t) = \Theta(t), \quad H(t) = \Psi(t) \tag{3.11}$$

is a realization of W. On the other hand, the condition is surely necessary since from (3.2) we have, for any fixed t_0,

$$W(t, \tau) = H(t)\Phi(t, t_o)\Phi(t_o, \tau)G(\tau) \tag{3.12}$$

so that we may set

$$\begin{aligned} \Psi(t) &= H(t)\Phi(t, t_o) \\ \Theta(t) &= \Phi(t_o, t)G(t). \end{aligned} \tag{3.13}$$

Q.E.D.

Suppose now that $\Psi(\cdot)$ and $\Theta(\cdot)$ are functions as in Theorem 8 and consider the C-impulse response defined by

$$\begin{aligned} W_C(t, \tau) &= \Psi(t)\Theta(\tau), \quad t \geqq \tau, \\ &= 0 \qquad\quad\;\;, \quad t < \tau. \end{aligned} \tag{3.14}$$

Then Theorem 8 shows that the function

$$\begin{aligned} Z(t, \tau) &= 0, \qquad\qquad t > \tau \\ &= \Psi(t)\Theta(\tau), \quad t \leqq \tau \end{aligned}$$

is identical with the A-impulse response associated with the realization (3.11). Moreover, if W_C is the C-impulse response of a dynamical system (2.1), then the realization (3.14) is an immediate consequence of Theorem 8 and relation (3.5). Thus, we have

COROLLARY TO THEOREM 8. *A C-impulse response is realizable by means of a dynamical system (2.1) if and only if relation (3.14) holds. An analogous result holds for the A-impulse response.*

Now consider

DEFINITION 10. The <u>weighting pattern</u> W in (3.10) is in <u>reduced form</u> if the columns of $\Psi(\cdot)$ and the rows of $\Theta(\cdot)$ are both linearly independent sets of functions on the real line. Otherwise W is <u>reducible</u>.

DEFINITION 11. If W in (3.10) is in reduced form, the number of columns of Ψ (= number of rows of Θ) is called the <u>order</u> of W.

DEFINITION 12. A <u>global realization</u> (2.1) of W in (3.10) is in <u>reduced form</u> if the dimension of the state space is the same as the order of W.

We now have

THEOREM 9. <u>A reduced form realization</u> (2.1) <u>of a given weighting pattern</u> (3.10) <u>always exists</u>.

<u>Proof</u>. If W in (3.10) is in reduced form, then the realization

$$\dot{x}(t) = \Theta(t)u(t)$$
$$y(t) = \Psi(t)x(t)$$

obviously has a state space dimension equal to the number of rows of Θ. Hence to prove the theorem, we need to show that any weighing pattern can be put into reduced form. To do this, consider an $r \times p$ reducible weighting pattern

$$W(t, \tau) = \alpha(t)\beta(\tau) \tag{3.15}$$

in which $\alpha(\cdot)$ is $r \times (n+q)$, $\beta(\cdot)$ is $(n+q) \times p$ and the row rank of $\beta(\cdot)$ is n. Without loss of generality, assume the first q rows of β to be linearly dependent on the remaining n rows. Then, by performing elementary row operations on β [12] one can construct a transformation whose insertion into the expression (3.15) allows the latter to be written as

(3.16) $$W(t, \tau) = \hat{\alpha}(t)\hat{\beta}(\tau)$$

where the first q rows of $\hat{\beta}$, namely $\{\hat{\beta}_q\}$, are $\underline{0}$. It then follows that the first q columns $\{\hat{\alpha}_q\}$ of $\hat{\alpha}$ are superfluous, and if we partition $\hat{\alpha}$ and $\hat{\beta}$ into $[\{\hat{\alpha}_q\}|A]$ and $\left[\frac{\{\hat{\beta}_q\}}{B}\right]$ respectively, then (3.16) can be written as

(3.17) $$W(t, \tau) = A(t)B(\tau).$$

If the columns of $A(\cdot)$ are linearly independent on the real line, then (3.17) is in reduced form. If, on the other hand, the column rank of $A(\cdot)$ is $s < n$, we can perform elementary column operations on $A(\cdot)$ and from this construct a linear transformation which would enable us to write (3.17) in the form

$$W(t, \tau) = \hat{A}(t)\hat{B}(t)$$

where $n-s$ columns of $\hat{A}(\cdot)$ are $\underline{0}$ thus rendering $n-s$ rows of $\hat{B}(\cdot)$ superfluous. By strict analogy with the the previous "partitioning" argument, we finally end up with

$$W(t, \tau) = \Psi(t)\Theta(\tau)$$

where the s columns of $\Psi(\cdot)$ and s rows of $\Theta(\cdot)$ are linearly independent functions on the real line. Q.E.D.

THEOREM 10. (i) <u>A global reduced-form realization</u> (2.1) <u>of a given reduced-form weighting pattern</u> (3.10) <u>has the lowest dimension of all global realizations,</u> <u>and</u> (ii) <u>conversely,</u> <u>all minimum-dimension realizations</u> (2.1) <u>of a reduced-form weighting pattern</u> (3.10) <u>are in reduced form</u>.

Proof. (i) Let n be the order of the weighting pattern and assume there exists a realization (2.1) whose state space is of lower dimension than n. Then the number of rows $\Phi(t_0, \cdot)G(\cdot)$ must be less than n and hence the order of the reduced form weighting pattern for this system is also less than n, which is a contradiction.

(ii) Suppose there is a minimum-dimension realization which is not in reduced form. Upon reduction, its dimension must be less than n, which contradicts the minimality of n. Q.E.D.

3. Uniqueness of the Realization.

A realization of W, even if in reduced form, is never unique for the following reason. According to the axiomatic definition of a system, the state is always an abstract quantity. Therefore replacing a given set of state variables by another equivalent set must clearly not affect input-output relations. It is easy to verify this general fact in the present case.

Let $T(t)$, $-\infty < t < \infty$, be a family of nonsingular linear transformations such that the function $T(\cdot)$ is continuously differentiable. Then we can set up a diffeomorphism (i.e., a 1×1 bidifferentiable correspondence) between $\mathcal{T} \times \Sigma$ and $\hat{\mathcal{T}} \times \hat{\Sigma}$ by writing

$$(\hat{t}, \hat{x}) = (t, T(t)x). \tag{3.18}$$

Then the matrices $F(t)$, $G(t)$, $H(t)$ in (2.1) are to be replaced respectively by

$$\hat{F}(t) = \dot{T}(t)T^{-1}(t) + T(t)F(t)T^{-1}(t)$$
$$\hat{G}(t) = T(t)G(t)$$
$$\hat{H}(t) = T^{-1}(t)H(t).$$

$$\hat{\Phi}(t, \tau) = T(t)\Phi(t, \tau)T^{-1}(\tau).$$

If W is the weighting pattern associated with $\hat{F}$, $\hat{G}$, $\hat{H}$, then

$$\begin{aligned}\hat{W}(t, \tau) &= \hat{H}(t)\hat{\Phi}(t, \tau)\hat{G}(\tau) \\ &= [H(t)T^{-1}(t)] \cdot [T(t)\Phi(t, \tau)T^{-1}(\tau)] \cdot [T(\tau)G(\tau)] \\ &= H(t)\Phi(t, \tau)G(\tau) \\ &= W(t, \tau).\end{aligned}$$

Thus in any case a <u>realization of</u> W <u>is unique only up to a diffeomorphism</u> (3.15). In fact, it can be shown that any two reduced form realizations of a given weighting pattern <u>must</u> be related by such a diffeomorphism [13].

3.5. <u>Weighting Patterns and the Concepts of Controllability and Observability</u>.

In this section, we explore the controllability and observability properties of global reduced-form realizations of weighting patterns. The results obtained will allow us to obtain a specific link between the weighting pattern of a system and the system's canonical structure. For this purpose, it will be convenient to list certain appropriate properties of weighting patterns and relate these properties to the concepts of controllability and observability.

We assume first, without loss of generality, that in all weighting patterns (3.10), the columns of $\Psi(\cdot)$ and rows of $\Theta(\cdot)$ are linearly independent functions on the entire real line (i.e., the weighting patterns are in reduced form when the entire t, τ plane is considered). Now consider the following possible properties of these weighting patterns.

<u>P1</u>. The rows of $\Theta(\cdot)$ are linearly independent functions on every interval of the real line.

<u>P2</u>. Same as P1 with "rows of Θ" replaced by "columns of Ψ".

<u>P3</u>. The rows of $\Theta(\cdot)$ are linearly independent functions on every semi-infinite interval with $+\infty$ as end point.

<u>P4</u>. Same as P3 with "rows of Θ" replaced by "columns of Ψ".

<u>P5</u>. Same as P3 with "$+\infty$" replaced by "$-\infty$".

<u>P6</u>. " " P4 " " " " ".

<u>P7</u>. (<u>a</u>) There exists an isolated, minimum length finite interval $I = [a, b]$, the closest such interval to $+\infty$, such that the rows of $\Theta(\cdot)$ are linearly independent functions over it.

<u>P7</u>. (<u>b</u>) Same as P7(a) except I is an ϵ-interval $(b^-, b]$.

<u>P8.</u> <u>a</u>, <u>b</u>. Same as P7 a, b respectively except "$+\infty$" is replaced by "$-\infty$" and "$(b^-, b]$" is replaced by "$[a, a^+)$".

<u>P9</u>. (<u>a</u>) There exists an isolated, minimum length finite interval I $I = [c, d]$, the closest such interval to $+\infty$, such that the columns of $\Psi(\cdot)$ are linearly independent functions over it.

<u>P9</u>. (b) Same as P9(a) except I is an ϵ-interval $(d^-, d]$.

<u>P.10</u>. <u>a</u>, <u>b</u>. Same as P9a, b respectively except "$+\infty$" is replaced by "$-\infty$", and "$(d^-, d]$" is replaced by "$[c, c^+)$".

Applications of Theorems 1-6 to P1-P10 yields the following statements in the form of a theorem about the reduced form realizations of the associated weighting patterns.

THEOREM 11. <u>Given a weighting pattern</u> (3.10) <u>and a corresponding reduced-form realization</u> (3.1). <u>Then</u>

(i) P1 <==> the system is differentially controllable $\forall t$

(ii) P2 <==> " " " differentially observable $\forall t$

(iii) P3 <==> " " " C-controllable $\forall t$

(iv) P4 <==> " " " A-observable $\forall t$

(v) P5 <==> " " " A-controllable $\forall t$

(vi) P6 <==> " " " C-observable $\forall t$

(vii)(a) P7(a) => " " " C-controllable $\forall t \leq a$
(<== if, in addition, the system is not C-controllable for any $t > a$)

(vii)(b) Same as (vii)(a) except "$\forall t \leq a$"$\rightarrow$"$\forall t < b$", and "$t > a$" $\rightarrow$ "$t > b$".*

(viii)(a) P8(a) ==> the system is A-controllable $\forall t \geq b$
(<== if, in addition, the system is not A-controllable for any $t < b$)

(viii)(b) Same as (viii)(a) except "$\forall t \geq b$"$\rightarrow$"$\forall t > a$", and "$t < b$"$\rightarrow$"$t < a$".

(ix)(a) P9(a) ==> the system is A-observable $\forall t \leq c$
(<== if, in addition, the system is not A-observable for any $t > c$)

(ix)(b) Same as (ix)(a) except "$\forall t \leq c$"$\rightarrow$"$\forall t < d$", and "$t > c$" $\rightarrow$ "$t > d$".

(x)(a) P10(a) ==> the system is C-observable $\forall t \geq d$
(<== if, in addition, the system is not C-observable for any $t < d$)

(x)(b) Same as (x)(a) except "$\forall t \geq d$"$\rightarrow$"$\forall t > c$", and "$t < d$" $\rightarrow$ "$t < c$".

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* The arrow "$\rightarrow$" denotes the phrase "is replaced by".

Now consider

THEOREM 12. <u>Given a system</u> (2.1) <u>with a canonical decomposition</u> (2.14) <u>of type</u> I, II, III, <u>or</u> IV. <u>Then, for any chosen type, if the dimension of the simultaneously controllable and observable part of the decomposition is constant</u> (<u>so that this part is a subsystem</u>), <u>the weighting pattern of the system,</u> $W(t, \tau)$, <u>is given by</u>

$$W(t, \tau) = H^b(t)\Phi^{bb}(t, \tau)G^b(\tau) \tag{3.19}$$

<u>where</u> Φ^{bb} <u>is the transition matrix corresponding to</u> F^{bb}. <u>Moreover, the</u> "b" <u>part of the given canonical decomposition is a reduced form realization of the weighting pattern</u>.

<u>Proof</u>. The first part of the theorem follows from formula (3.12) with substitution of (2.14). To prove the second part of the theorem, assume that the "b" part of the appropriate type canonical decomposition is <u>not</u> a reduced form realization of the system's weighting pattern. In that case, if n is the order of W, the realization of part (b) must be of higher dimension than n, which means that either the columns of $H^b(\cdot)\Phi^{bb}(\cdot, t_o)$ or the rows of $\Phi^{bb}(t_o, \cdot)G^b(\cdot)$ are linearly dependent functions on the real line. But from Theorems 1-4, this means that the realization cannot be simultaneously controllable and observable (using the appropriate definitions corresponding to the given type of decomposition) which is a contradiction. Q.E.D.

We now ask two questions:

(i) What properties of a weighting pattern imply that a given realization can have a canonical decomposition of a specified type in which the simultaneously controllable and observable part has constant dimension?

(ii) Conversely, if we know that one or more specified type canonical decompositions of a given system have a "b" part which is of constant dimension, what does this imply about the weighting pattern?

An answer to these questions is provided by table I and the comment which follows it. The right hand column gives the possible sets of the different types of canonical decompositions of a system in which the dimension of the simultaneously controllable and observable part is constant. The left hand column plus the comment below Table I indicate the corresponding properties of the weighting pattern.

TABLE I

WEIGHTING PATTERN PROPERTIES	TYPE OF DECOMPOSITION FOR WHICH n_b IS CONSTANT
P1, P2	I, II, III, IV
P1, P4	II, III
P1, P6	I, IV
P2, P3	I, III
P2, P5	II, IV
P3, P4	III
P3, P6	I
P4, P5	II
P5, P6	IV

Completion of table I is accomplished by noting that the right hand column is unchanged if, in the left hand column, we make the substitutions

$$P_1 \rightarrow P_3 + P_5$$

$$P_2 \rightarrow P_4 + P_6.$$

From Theorem 12 and table I we obtain (see [3])

COROLLARY to THEOREM 12. A reduced-form realization of a weighting pattern having one or more of the sets of properties listed in table I must have one or more of the properties below.

(i) Complete C-controllability and complete C-observability

(ii) complete A-controllability and complete A-observability

(iii) complete C-controllability and complete A-observability

(iv) complete A-controllability and complete C-observability.

3.6. The Role of C and A Impulse Responses.

In the previous section, we discussed the relationship of the weighting pattern of a system to the canonical decompositions of the system. Similar (though more restricted) results can be obtained in terms of the C and A impulse responses of the system. An important result, implied by Theorem 11, is given by

THEOREM 13. Consider a reduced-form weighting pattern (3.10), and a corresponding reduced-form realization (2.1).

(i) The intersection of the intervals (in τ) of C-controllability and C-observability of (2.1) can be determined from the A-impulse response of the system.

(ii) The intersection of the intervals (in τ) of A-controllability and A-observability of (2.1) can be determined from the C-impulse response of the system.

Proof. (i) If the intersection in (i) above is not empty, Theorem 13 implies that for every t-interval I_t on which the columns of $\Psi(\cdot)$ in (3.10) are linearly independent functions, there exists a τ-interval I_τ, such that $\tau > t$, $\forall \tau \in I_\tau$, $\forall t \in I_t$, on which the $\Theta(\cdot)$ are linearly independent functions. The first part of the theorem then follows from the fact that the A-impulse response $W_A(t, \tau)$ essentially coincides with the weighting pattern $W(t, \tau)$ for $t \leq \tau$.

(ii) Same as above with Ψ and Θ interchanged, t and τ interchanged, and A-impulse response replaced by C-impulse response. Q.E.D.

In general, the C- (A-) impulse response yields no information about intervals of simultaneous C- (A-) controllability and C- (A-) observability. In special cases, however, complete information is obtainable. It is clear, for example, that if the respective "b" parts of a type I and type II canonical decomposition of a system are identical (so that all four decompositions have identical "b" parts), then Theorem 12 can be stated in terms of C- or A-impulse responses as well as in terms of the weighting pattern. Hence, an essential hypothesis for the result stated by Kalman [1], [9], linking (C-) impulse responses to the "b" part of a (type I) canonical decomposition is that this "b" part is also defined by a type II decomposition of the system (see [3]).

To illustrate the above statements, consider the one-dimensional system

$$\frac{dx}{dt} = g(t)u(t) \tag{3.18}$$

$$y(t) = h(t)x(t).$$

Suppose $g(\cdot)$ and $h(\cdot)$ are defined as in Figure 3; i.e., they are unity over one semi-infinite interval, zero over another, and make a smooth transition between the levels o and 1. In addition their supports are disjoint.

The reader can easily verify the following facts.

(i) The system (3.18) is completely C-controllable and completely C-observable.

(ii) The system is nowhere simultaneously A-controllable and A-observable.

(iii) The C-impulse response of (3.18) is identically zero.

(iv) The A-impulse response $W_A(t, \tau)$ is non-zero if $t < 0$ and $\tau > 0$.

If the roles played by the functions $g(\cdot)$ and $h(\cdot)$ are reversed in (3.18), the above statements hold with "C" and "A" interchanged and "t" and "τ" interchanged.

4. Adjoint and Dual Systems.

Theorem 1-6 give ample evidence of the existence of a "duality between the concepts of controllability and observability. The purpose of this section is to give precise meaning to the preceding statement. In order to do so, we define and discuss the concepts of adjoint and dual systems.

4.1. Definition of Adjoint Systems.

We give the definition indicated in the title and relate it to the classical concept of the adjoint of a linear differential operator.

DEFINITION 13. Let S be a reduced-form linear differential system (2.1) with weighting pattern $W(t, \tau)$. Then the adjoint system S^*, with associated weighting pattern $W^*(t, \tau)$, is defined by the following properties.

(i) The systems S and S^* have the same time set with the same ordering.

(ii) The product of the transposed transition matrix for S with the transition matrix for S^* is the identity.

(iii) $W^*(t, \tau) = - W'(\tau, t)$

where the "prime" denotes transpose.

DEFINITION 14. Two systems S and S^* are input-output adjoints of each other if properties (i) and (iii) of Definition 13 hold.

4.2. Relation to Classical Adjoint.

Consider the homogeneous n-dimensional vector differential equations

$$\frac{dx}{dt} = F(t)x \tag{4.1}$$

$$\frac{d\hat{x}}{dt} = - F'(t)\hat{x}. \tag{4.2}$$

It is readily demonstrated [5] that the product of a fundamental matrix solution of (4.1) with the transpose of that for (4.2) is a constant matrix. In fact, if $\Phi(t, \tau)$ is the transition matrix for (4.1), then the transition matrix for (4.2) is $\Phi'^{-1}(t, \tau)$. This essentially defines (4.1) and (4.2) as being adjoint equations [5].

Now suppose we add to (4.1) and (4.2) the respective r-dimensional vector equations

$$y(t) = H(t)x(t) \tag{4.3}$$

$$\hat{y}(t) = H(t)\hat{x}(t) \tag{4.4}$$

and proceed to eliminate the state variables in each case so that we are left with homogeneous n^{th}-order vector differential equations in y and $\hat{y}$ respectively, as below

$$L\{y\} = 0 \tag{4.5}$$

$$L^*\{\hat{y}\} = 0. \tag{4.6}$$

Then the operators L and L^* must be formal adjoints of each other, i.e., if

$$L\{\cdot\} = \sum_{i=0}^{n} A_i(t) \frac{d^i}{dt^i} \{\cdot\}$$

where the $\{A_i\}$ are $r \times r$ matrices, then

$$L^*\{\cdot\} = \sum_{i=0}^{n} (-1)^i \frac{d^i}{dt^i} \{A_i(t)\cdot\}.$$

The equations (4.5, 6) are also called adjoint equations.

The adjoint equation to (4.5) arises naturally from another point of view, namely, in the search for (vector) integrating factors for (4.5). That is, every solution of the adjoint equation (4.6) is an integrating factor for (4.5) and this fact provides the basis for the (vector forms of the) Lagrange identity and Green's formula (see [5]).

To connect the classical theory to Definition 13 we point out the simple fact that in the case of zero input, the system, S and S* in Definition 13 are adjoint in the classical sense.

The need for Definition 14 is shown by the following. Consider two systems S_1, S_2 whose respective input-output relations can be described by vector differential equations of the form

$$L_1\{y_1\} = u_1 \tag{4.7}$$

$$L_2\{y_2\} = u_2 \tag{4.8}$$

where

$$\dim y_1 = \dim y_2$$

$$\dim u_1 = \dim u_2$$

and L_1 is the formal adjoint of L_2. If the time scale for S_1 and S_2 is the same, then it is readily shown that S_1 and S_2 are input-output adjoints according to Definition 14. Also, if $u_1 = u_2 = 0$, then (4.7, 8) are adjoint equations. However, because of the nonuniqueness of realizations of input-output relations (see section 3.4), the systems S_1 and S_2 may not be adjoint with respect to their state variables. Hence, Definition 14 $\not\Rightarrow$ Definition 13.

4.3. Definition of Dual Systems.

DEFINITION 15. Let S be a system as in Definition 13. The dual system $\tilde{S}$, with associated weighting pattern $\tilde{W}(t, \tau)$, is defined by the following properties.

(i) The time sets of S and $\tilde{S}$ are the same but are oppositely ordered.

(ii) If we reverse the time ordering in $\tilde{S}$ and take the product of the transposed transition matrix of the resulting system with the transition matrix of S, we obtain the identity.

(iii) $\tilde{W}(t, \tau) = W'(\tau, t)$

where the "prime" denotes transpose.

DEFINITION 16. Two systems S and $\tilde{S}$ are <u>input-output duals</u> of each other if properties (i) and (iii) of Definition 15 hold.

4.4. <u>Differential Equation Representation of Adjoint and Dual Systems</u>.

We explicitly show in this section that the dual of a given system is obtained from the adjoint by reversing the direction of time flow in the latter.

THEOREM 14. <u>Consider a reduced-form system</u> S <u>given by</u> (2.1). <u>Then the adjoint system</u> S* <u>is given by</u>

$$\frac{d\hat{x}}{dt} = - F'(t)\hat{x} \mp H'(t)\hat{u}(t)$$

$$\hat{y}(t) = \pm G'(t)\hat{x}(t) \tag{4.9}$$

<u>where</u> $\hat{u}$ <u>is the</u> (<u>r</u>-<u>dimensional</u>) <u>input and</u> y <u>is the</u> (<u>p</u>-<u>dimensional</u>) <u>output</u>.

<u>Proof</u>. (i) Let $\Phi(t, \tau)$ be the transition matrix for (2.1); let $\hat{\Phi}(t, \tau)$ be likewise for (4.9). Then, since (2.1) and (4.9) are real systems

$$\hat{\Phi}(t, \tau) = \Phi'^{-1}(t, \tau)$$

and therefore $\Phi'(t, \tau)\hat{\Phi}(t, \tau) = I$.

(ii) The weighting pattern of (2.1) is given by

$$W(t, \tau) = H(t)\Phi(t, \tau)G(\tau).$$

From Definition 14, the weighting pattern for S^* must be

$$\hat{W}(t, \tau) = - G'(t)\hat{\Phi}(t, \tau)H'(\tau).$$

Equation (4.9) is a realization of $\hat{W}$ and since (i) specifies the transition matrix, the realization is unique modulo a sign combination for u and y as shown. Q.E.D.

THEOREM 15. <u>Let</u> S <u>be a system as in Definition</u> 13. <u>Then the dual system</u> $\tilde{S}$ <u>is given by</u>

$$\begin{aligned} \frac{dz}{ds} &= F'(s)z + H'(s)\hat{u}(s) \\ \hat{y}(s) &= G'(s)z(s). \end{aligned} \tag{4.10}$$

<u>Proof</u>. We need merely see whether Definition 15 is satisfied.

(i) The time sets of S and $\tilde{S}$ are oppositely ordered by hypothesis.

(ii) If we replace s by t and change the sign of the derivative in (4.10), the result is essentially (4.9). Theorem 14 implies that the product of the transposed transition matrix of (4.9) with the transition matrix of (2.1) is the identity matrix.

(iii) If $\Phi(\cdot, \cdot)$ is the transition matrix function of (2.1), then $\Phi'(\cdot, \cdot)$ is the transition matrix function of (4.10). The weighting pattern $\tilde{W}$ for (4.10) is then given by

$$\tilde{W}(s, \tau) = G'(s)\Phi'(s, \tau)H'(\tau). \tag{4.11}$$

From (3.2) we see that

$$W(t, \tau) = \tilde{W}'(\tau, t).$$

Q.E.D.

The proof of Theorem 15 provides the following

COROLLARY. The dual of a given system (2.1) is the adjoint system with the time scale ordered in the opposite sense.

4.5. Dual Systems and Concepts of Controllability and Observability.

Our objective as stated at the beginning of section 4 is met by presenting the following theorems.

THEOREM 16. Consider a system (2.1), with its associated adjoint (4.9). If the system (2.1) is C[A] controllable (or C[A] observable) at $t = t_o$, then the adjoint system (4.9) is A[C] observable (or A[C] controllable) at $t = t_o$.

Proof. The analogies involved are such that it is only necessary to prove that C-controllability of (2.1) $\Longleftrightarrow$ A-observability of (4.9).

If the system (2.1) is C-controllable at $t = t_o$, then by Theorem 1 there exists $t_1 > t_o$ such that the columns of the matrix $G'(\cdot)\Phi'(t_o, \cdot)$ are linearly independent functions over the interval $[t_o, t_1]$. But this implies A-observability of (4.9) at t_o. The converse holds by reversibility of the argument. Q.E.D.

THEOREM 17. Consider a system (2.1), with its associated dual (4.10). If the system (2.1) is C[A] controllable (or C[A] observable) at $t = t_o$, then the dual system (4.9) is C[A] observable (or C[A] controllable) at $s = - t_o$.

Proof. Follows from Theorem 16 plus the corollary to Theorem 15.

5. Time-Domain Input-Output Relations - The Input-Output Differential Equation.

In section 3 we discussed the role of the weighting pattern in the input-output representation of linear systems. In certain situations the natural specification of the input-output characteristics of a system is via a differential equation. In this section we discuss the relationship of input-output differential equations to other modes of system representation.

5.1. Existence and Form.

THEOREM 18. Consider a system (2.1). For a class of inputs U whose components as sufficiently smooth, (2.1) implies the existence of a differential equation relating output y to input $u \in U$ of the form

$$Ly = Mu, \tag{5.1}$$

where

L = vector linear differential operator

M = " " " " ,

and in every such differential equation, the order of M is necessarily lower than that of L.

<u>Proof</u>. Differentiation of (2.1) yields

$$y - HGu = (HF + \dot{H})x.$$

Differentiating once again we have

$$\ddot{y} - HG\dot{u} - (H\dot{G} + HFG + 2HG)u$$
$$= (HF^2 + \dot{H}F + H\dot{F} + \ddot{H})x.$$

We continue in this manner until n derivatives have been taken, and then group the results from the zeroth derivative on down in matrix form, i.e.,

$$(5.2) \qquad \begin{bmatrix} y \\ y - HGu \\ \ddot{y} - HG\dot{u} - (2\dot{H}G + H\dot{G} + HFG)u \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} H \\ HF + \dot{H} \\ HF^2 + \dot{H}F + H\dot{F} + \ddot{H} \\ \vdots \\ \vdots \end{bmatrix} x.$$

If we consider the coefficient matrix on the right side of (5.2) at any time t_o, we can obtain a set of linearly dependent rows of dimension k, $2 \leq k \leq n + 1$, such that $k-1$ of these are linearly independent. We then merely express one of the row vectors in terms of the remaining $k-1$ vectors and rewrite this relationship using the left side of (5.2) (evaluated at t_o). The result is a relation involving y, u, and various of their derivatives all evaluated at t_o. Since t_o is an arbitrary point on the interval of definition of the system, then as long as the various derivatives of H, G, F are continuous, this procedure implies the existence of a differential equation relating y to u, with continuous (matrix) coefficients.

If the order of M is not $<$ the order of L, then the degree of smoothness of the elements of y (hence of x) cannot be greater than the degree of smoothness of the elements of u. It follows from (2.1) and its associated hypotheses that just the opposite is true. Q.E.D.

Remarks: 1. The differential equation obtained by the above technique will not be unique if $k < n + 1$ since there is then a choice of the row vectors used in the linear dependence argument. Similarly, even if $k = n + 1$, uniqueness can be destroyed by considering derivatives of (2.1) of order higher than n.

2. It should be emphasized that (5.1) is a well-defined ordinary differential equation only for input functions which are in the domain of M. However, it has become common practice, by introducing δ-functions, to regard (5.1) as an equation which has a meaning defined in accordance with the formal operations associated with δ-functions when the elements of the input u do not have the normally required smoothness properties.

5.2. Relationship of a System's Input-output Differential Equation to its Canonical Structure.

It is assumed in this discussion that the dimension numbers of those parts of a system's canonical decomposition (type I, II, III, or IV) labeled "b" and "d" in Theorem 7 are constants.

The main result is given by

THEOREM 19. (i) *The input-output differential equation of a given system* (2.1) *depends only on parts* "b" *and* "d" *of the system's canonical decomposition* (*type* I, II, III, *or* IV).

(ii) <u>If the dimension of the "b" part is zero, the input-output differential equation is a homogeneous equation in the output</u> y.

<u>Proof</u>. (i) Follows from the relation (see (2.14))

$$y(t) = H^b(t)x^b(t) + H^d(t)x^d(t).$$

(ii) Since $x^b(t) \equiv 0$, the input-output differential equation is essentially implied by the relations

$$\frac{dx^d}{dt} = F^{dd}(t)x^d$$
$$y(t) = H^d(t)x^d(t). \qquad \text{Q.E.D.}$$

Remark: It is obvious that the input-output differential equation determines the weighting pattern of a system. If the "d" part of the canonical decomposition is empty, then, under certain additional assumptions, it is easy to prove that the weighting pattern determines the input-output differential equation. (See section 5.5.)

5.3. <u>Relationship to the State-variable Differential Equation - Preliminaries</u>.

We now consider the problem of obtaining the input-output differential equation from a given state variable differential equation.

For simplicity we restrict ourselves to those systems such that

(i) the dimension numbers of all parts of any given canonical decomposition, with the exception of the "b" part, are zero.

(ii) The systems possess the property of differential observability on their intervals of definition. (See Definition 6.)

Consider the differentially observable system (2.1). As far as the input-output characteristics are concerned, we may replace (2.1) by the more convenient set of equations

$$\frac{d\hat{x}}{dt} = \Theta(t)u(t) \tag{5.3 (a)}$$

$$y(t) = \Psi(t)\hat{x}(t) \tag{5.3 (b)}$$

where $\Psi(\cdot)$ and $\Theta(\cdot)$ are as given in (3.13), and

$$\hat{x}(t) = \Phi(t_0, t)x(t) \tag{5.4}$$

so that (5.3) and (2.1) are related by a linear transformation on the state space as shown in (5.4).

If the (left) inverse of $\Psi(\cdot)$ in (5.3) were to exist, one could solve for $\hat{x}$ in (5.3)(b) and substitute the result into (5.3) (a) to obtain a differential equation relating y to u. Since $\Psi(\cdot)$ is an $r \times n$ matrix with $r \leq n$, the rank of Ψ can be n only if $r = n$, and the latter condition is necessary if such a trivial solution to the posed problem can be found.

Let us consider the more interesting case $r < n$. The (left) inverse of $\Psi(\cdot)$ does not exist, but if we differentiate (5.3) (b) and group the resulting equation with (5.3) (b), the rank of the coefficient matrix of the state vector will be larger than the rank of $\Psi(\cdot)$. The question arises as to the number of derivatives needed in order to solve for $\hat{x}$. Because of notational complexities, it is expedient to answer the preceding question by considering separately the two cases listed below.

Case 1: n is divisible by r

Case 2: n is not divisible by r.

5.4. Discussion of Case 1.

Let $\ell = \frac{n}{r}$.

We then have

THEOREM 20. Consider the differentially observable system (5.3) defined on an interval I. If n (the dimension of the state space) is divisible by r (the dimension of the output space) then the matrix

$$V_{\{\Psi\}}(t) = \text{col}\ (\Psi(t),\ \Psi^{(1)}(t),\ \ldots,\ \Psi^{(\ell-1)}(t)) \tag{5.5}$$

is nonsingular $\forall t \in I$.

Proof. By Theorem 6 the n columns of $\Psi(\cdot)$ must satisfy a vector differential equation defined on I. By applying the machinery used in the proof of Theorem 5, it is easily shown that such an equation exists of order ℓ. Then $V_{\{\Psi\}}(\cdot)$ in (5.5) is a fundamental matrix of the latter and is hence nonsingular on I. Q.E.D.

Consider now the first ℓ derivatives of (5.3) (b), given by

$$y^{(k)}(t) = \Psi^{(k)}(t)\hat{x}(t) + \sum_{j=0}^{k-1} \binom{k-1}{j} \left[\left.\frac{\partial^j W(t, \tau)}{\partial \tau^j}\right|_{\tau = t} + \left.\frac{\partial^j W(t, \tau)}{\partial t^j}\right|_{\tau = t} \right] u^{(k-j-1)}(t), \quad k = 1, \ldots, \ell \tag{5.6}$$

where $W(t, \tau)$ is the weighting pattern of (5.3) given by

$$W(t, \tau) = \Psi(t)\Theta(\tau). \tag{5.7}$$

Equation (5.6) is a set of ℓ r-dimensional vector equations. If we group (5.3) (b) with the first ℓ-1 r-dimensional vector sets together we get the equation

$$Y(t) = V_{\{\Psi\}}(t)\hat{x}(t) + U(t) \tag{5.8}$$

where

$$Y(t) = \text{col}\,\{y(t),\ y^{(1)}(t),\ \ldots,\ y^{(\ell-1)}(t)\}$$

$V_{\{\Psi\}}(t)$ is given by (5.5)

$$U(t) = \text{col}\,\{0,\ W(t,\tau)u(t),\ \ldots,\ \sum_{j=0}^{\ell-2} \binom{\ell-2}{j} \left[\left.\frac{\partial^j W(t,\tau)}{\partial\tau^j}\right|_{\tau=t} + \left.\frac{\partial^j W(t,\tau)}{\partial t^j}\right|_{\tau=t} \right] u^{(\ell-j-1)}(t)\}.$$

From Theorem 21, the inverse of $V_{\{\Psi\}}$ exists. Hence, we can solve for $\hat{x}$ in (5.8) and substitute the result into the $\ell^{\underline{th}}$ equation of (5.6) to yield the desired result

$$I_r y^{(\ell)}(t) - \Psi^{(\ell)}(t)V_{\{\Psi\}}^{-1}(t)Y(t) = -\Psi^{(\ell)}(t)V_{\{\Psi\}}^{-1}(t)U(t) + \sum_{j=0}^{\ell-1} \binom{\ell-1}{j} \left[\left.\frac{\partial^j W(t,\tau)}{\partial\tau^j}\right|_{\tau=t} + \left.\frac{\partial^j W(t,\tau)}{\partial t^j}\right|_{\tau=t} \right] u^{(\ell-j-1)}(t) \tag{5.9}$$

where

$$I_r = r \times r \text{ identity matrix}$$

$$V^{-1}_{\{\Psi\}}(t) = \text{inverse of } V_{\{\Psi\}}(t) \text{ in (5.5).}$$

Notes: (i) The homogeneous equation ((5.9) with $u \equiv 0$) is satisfied by the n columns of $\Psi(\cdot)$. In fact, the latter is a fundamental matrix for, and uniquely determines, the homogeneous version of (5.9).

(ii) If $r = 1$, $V_{\{\Psi\}}(\cdot)$ becomes the Wronskian matrix of the elements of $\Psi(\cdot)$.

5.5. Continued Discussion of Case 1 - The Relationship of the Input-Output Differential Equation to the Weighting Pattern.

We shall show first how to construct the weighting pattern of a system from its input-output differential equation (5.9) and secondly how to construct the input-output equation of a system from its weighting pattern (5.7).

We begin by writing (5.9) in the form

$$\sum_{i=0}^{\ell} A_i(t)y^{(i)} = \sum_{j=0}^{\ell-1} B_j(t)u^{(j)} \tag{5.10}$$

where the $\{A_i(t)\}$ are real $r \times r$ matrices, $A_\ell(t)$ = the identity matrix, and the $\{B_j(t)\}$ are real $r \times p$ matrices. If we define the operator on the left hand side of (5.10) as

$$L_t\{\cdot\} = \sum_{i=0}^{\ell} A_i(t) \frac{d^i}{dt^i}\{\cdot\}$$

then the formal adjoint of L_t, denoted by L_t^*, is given by

$$L_t^*\{\cdot\} = \sum_{i=0}^{\ell} (-1)^i \frac{d^i}{dt^i} \{A_i'(t)\cdot\} \tag{5.11}$$

where the "prime" indicates transpose.

Let Ψ^* be a fundamental matrix solution of the homogeneous solution

$$L_t^*\{v\} = 0 \quad (v = \text{r-dimensional row vector})$$

such that the function P defined by

$$P(t, \tau) = \Psi(t)\Psi^*(\tau) \tag{5.12}$$

has the property

$$\left.\frac{\partial^i P(t, \tau)}{\partial t^i}\right|_{\tau = t} = \begin{cases} 0, & i = 0,1, \ldots, \ell - 2 \\ A_\ell^{-1}(t) = I_r, & i = \ell - 1 \end{cases} \tag{5.13}$$

It is easy to show, by a straightforward extension of the scalar case [2], that $P(t, \tau)$ is the weighting pattern for the system whose input-output relation is given by

$$L_t\{y\} = v, \tag{5.14}$$

i.e., if all initial conditions are zero at $t = t_o$, we have

$$y(t) = \int_{t_o}^{t} P(t, \tau)v(\tau)d\tau. \tag{5.15}$$

If we formally identify (5.14) with (5.10), we can rewrite (5.15) as

$$y(t) = \int_{t_o}^{t} P(t, \tau)M_\tau\{u(\tau)\}d\tau \tag{5.16}$$

where

$$M_\tau\{\cdot\} = \sum_{j=0}^{\ell-1} B_j(\tau) \frac{d^j}{d\tau^j} \{\cdot\}. \tag{5.17}$$

Consecutive integration of (5.16) by parts provides the final result

$$y(t) = \int_{t_o}^{t} M_\tau^*\{P(t, \tau)\}u(\tau)d\tau \tag{5.18}$$

where M_τ^* is the formal adjoint of M_τ, i.e.,

$$M_\tau^*\{\cdot\} = \sum_{j=0}^{\ell-1} (-1)^j \frac{d^j}{d\tau^j} \{B_j'(\tau)\cdot\}. \tag{5.19}$$

Comparing (5.18) with (3.3), we have that

$$W(t, \tau) = M_\tau^*\{P(t, \tau)\}. \tag{5.20}$$

which is a well known result in the scalar case [2].

The inverse problem of obtaining an input-output differential equation from a given weighting pattern is easily handled by first realizing the weighting pattern as a system (5.3) and the proceeding as in section 5.4.

5.6. <u>Discussion of Case</u> 2 (n <u>not divisible by</u> r).

To obtain the input-output differential equation from the state-variable equation in this case, the procedure is as follows. Differentiate (5.3) (b) as before to obtain (5.4). The highest needed value of k is now the integer succeeding the number $\frac{n}{r}$. Let this integer be q and let $m = n - r\cdot q$ $(m < r)$. The first $q-1$ derivatives of (5.3) (b) are of the form

$$
\begin{aligned}
y(t) &= \Psi(t)\hat{x}(t) \\
y^{(1)}(t) &= \Psi^{(1)}(t)\hat{x}(t) + \Theta(t)u(t) \\
&\;\;\vdots \qquad\qquad \vdots \\
y^{(q-1)}(t) &= \Psi^{(q-1)}(t)\hat{x}(t) + (\qquad).
\end{aligned}
\tag{5.21}
$$

Since the columns of $\Psi(\cdot)$ satisfy a differential equation, the coefficient matrix of $\hat{x}$ in (5.21) is always of maximal rank. Hence choose the first m components of the last equation in (5.21) and differentiate. The resulting equations have the form

$$
\begin{aligned}
y_1^{(q)}(t) &= \Psi_{R_1}^{(q)}(t)\hat{x}(t) + (\qquad) \\
&\;\;\vdots \qquad\qquad \vdots \\
y_m^{(q)}(t) &= \Psi_{R_m}^{(q)}(t)\hat{x}(t) + (\qquad)
\end{aligned}
\tag{5.22}
$$

where $\Psi_{R_i}(\cdot)$ is the i^{th} row of the matrix $\Psi(\cdot)$.

If we regard (5.21, 22) as a set, the coefficient matrix of $\hat{x}$ is nonsingular so that $\hat{x}$ can be solved for. The result is substituted into the equations

$$
(5.23)\qquad
\begin{aligned}
y_{m+1}^{(q)}(t) &= \Psi_{R_{m+1}}^{(q)}(t)\hat{x}(t) + (\qquad) \\
&\ \vdots \\
y_r^{(q)} &= \Psi_{R_r}^{(q)}(t)\hat{x}(t) + (\qquad) \\
y_1^{(q+1)}(t) &= \Psi_{R_1}^{(q+1)}(t)\hat{x}(t) + (\qquad) \\
&\ \vdots \\
y_m^{(q+1)}(t) &= \Psi_{R_M}^{(q+1)}(t)\hat{x}(t) + (\qquad).
\end{aligned}
$$

We then have a set of r differential equations relating the variables $y_1, \ldots, y_r$ to $u_1, \ldots, u_p$. Of these equations, $r - m$ are of order q, the rest are of order $q+1$. The total number of linearly independent solutions is therefore $(r-m)\cdot q + m(q\cdot 1) = n$. We note that since the columns of $\Psi(\cdot)$ are solutions of the associated homogeneous equation, they must form a basis for the solution space of the latter.

We now discuss the problem of obtaining the weighting pattern from a given input-output differential equation. Since r doesn't divide n we cannot put (5.23)(with $\hat{x}$ substituted for) into the form (5.10) and have the determinant of the coefficient of the highest derivative $\neq 0$. Hence, given an input-output differential equation of the form (5.10) with $\det A_{q+1}(t) = 0$, we first check if the given equation is equivalent to (5.23) (with $\hat{x}$ substituted for). If it is, the weighting pattern is obtained as follows.

Let

$$z_1 = \begin{bmatrix} y_{m+1} \\ \cdot \\ \cdot \\ \cdot \\ y_r \end{bmatrix}$$

$$z_2 = z_1'$$

$$\vdots$$

$$z_q = z_{q-1}'$$

$$z_{q+1} = g(\Psi_{R_{m+1}}, \ldots, \Psi_{R_r}, y, y^{(1)}, \ldots, y^{(q-1)}, y_1^{(q)}, \ldots, y_m^{(q)}, u_1, \ldots, u_p^{(q)})$$

(5.24)

$$z_{q+2} = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{bmatrix}$$

$$z_{q+3} = z_{q+2}'$$

$$\vdots$$

$$z_{2q+2} = z_{2q+1}'$$

$$z_{2q+3} = f(\Psi_{R_1}, \ldots, \Psi_{R_m}, y, y^{(1)}, \ldots, y^{(q-1)}, \ldots, y_m^{(q)}, u_1, \ldots, u_p^{(q)}).$$

Equation (5.24) can be written as

$$(5.25) \qquad \begin{bmatrix} \frac{dz_1}{dt} \\ \cdot \\ \cdot \\ \cdot \\ \frac{dz_{2p+3}}{dt} \end{bmatrix} = A(t) \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_{2p+3} \end{bmatrix} + \tilde{G}(t)u(t).$$

Now, if $\tilde{\Phi}$ is the transition matrix for (5.25), the associated weighting pattern is

$$\tilde{W}(t, \tau) = \tilde{\Phi}(t, \tau)\tilde{G}(\tau).$$

The solution of (5.25) is (with zero initial conditions at $t = t_o$)

$$(5.26) \qquad \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_{2p+3} \end{bmatrix} = \int_{t_o}^{t} \tilde{W}(t, \tau)u(\tau)d\tau.$$

Since we are interested only in $y_1, \ldots, y_r$ (i.e., z_{p+2}, z_1), we simply eliminate all the rows of $\tilde{W}$ which don't correspond to z_{p+2}, z_1. The remaining matrix is the weighting pattern for the original differential equation.

6. Conclusion.

This paper has presented a comprehensive exposition of certain topics in linear system theory in which both new and old results have been incorporated within a single framework. The introduction of anti-causal systems into the theory has allowed meaningful extension of previous results and has aided in clarifying and solving some well known problems.

Although we have concentrated exclusively on time-domain representations for linear systems the concepts discussed here play a significant role in transform-domain analysis of linear systems. Specifically, the concepts of C and A impulse responses, adjoint systems, and dual systems arise quite naturally in the development of the system function approach to input-output analysis of linear systems [11].

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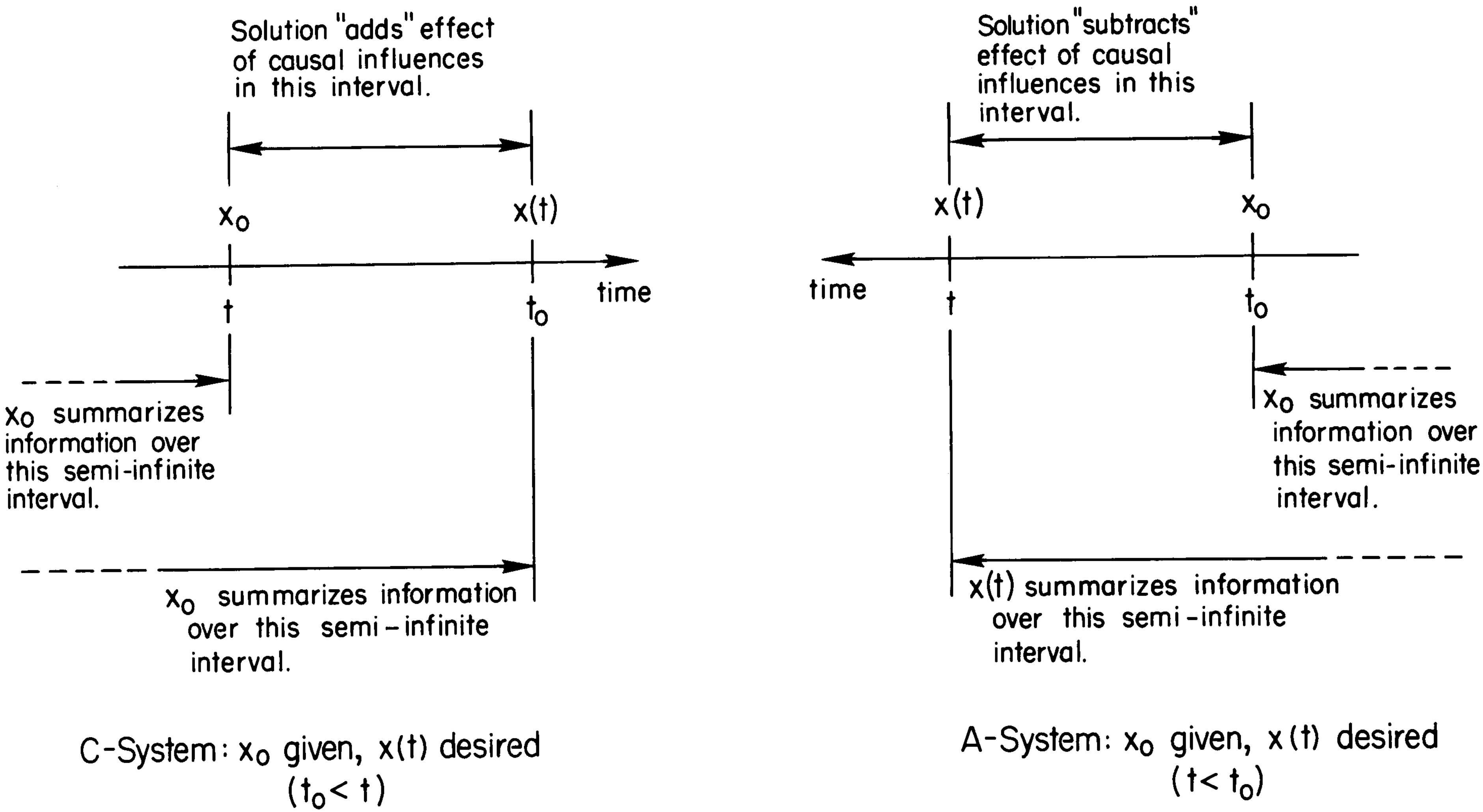

Figure 1 - C and A System Interpretations of Solution of Differential Equation

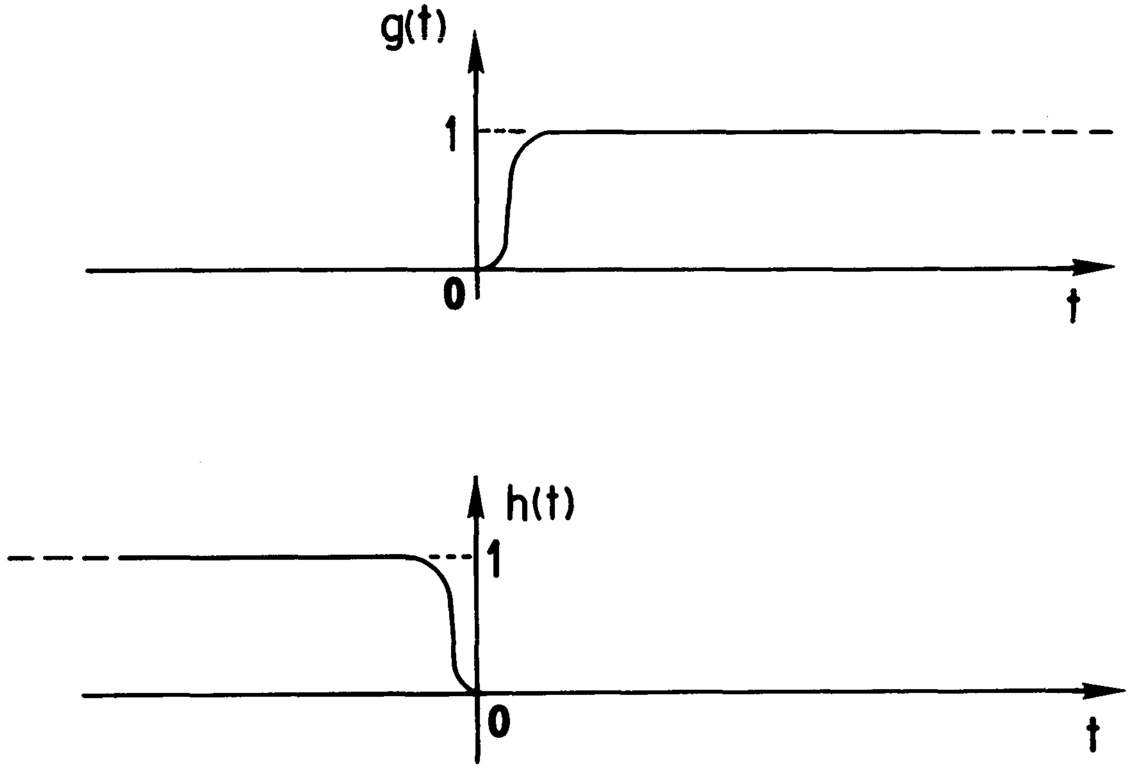

Figure 3 - The functions g(·) and h(·) in (3.18).

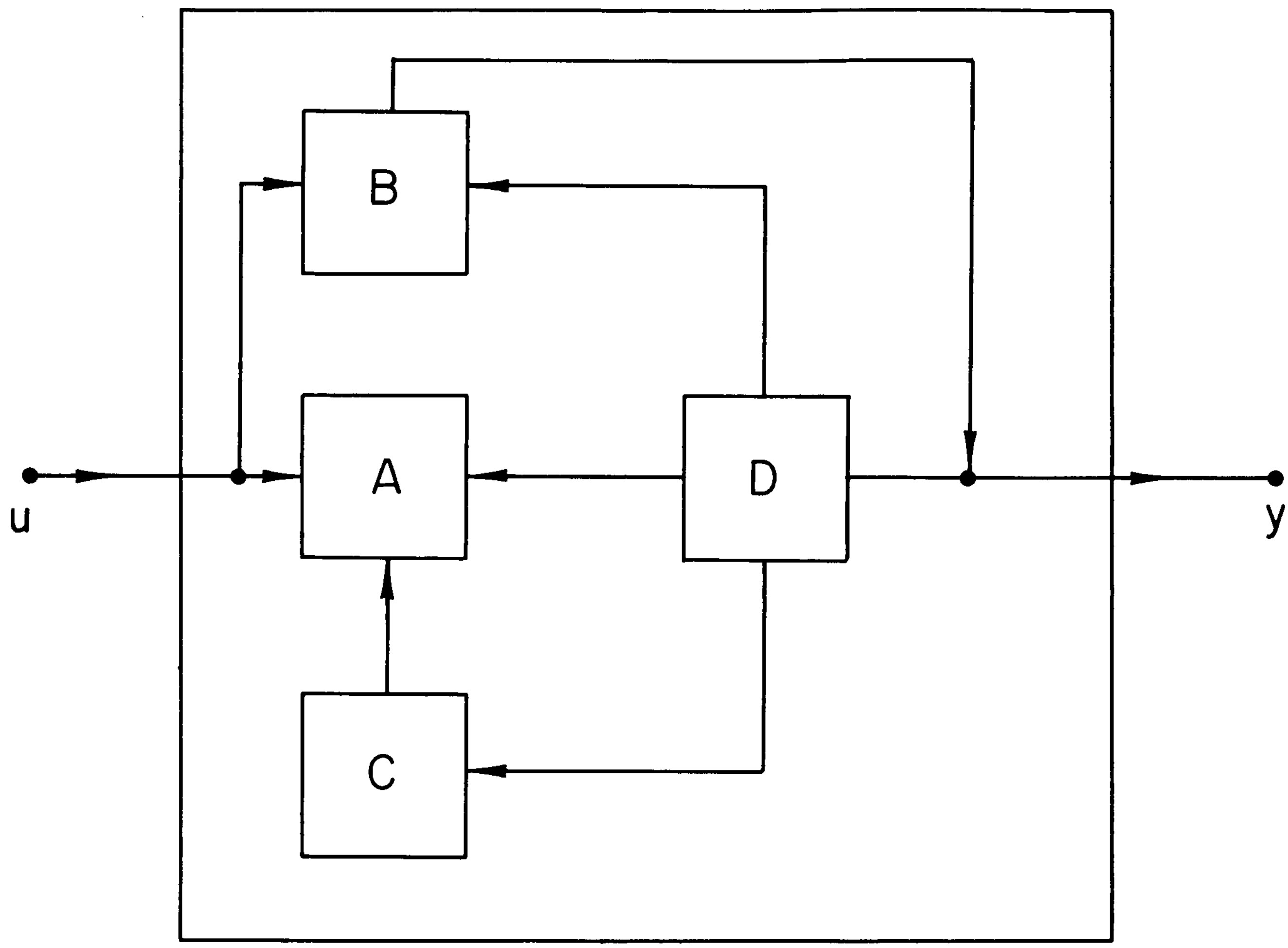

Figure 2 - The Canonical Structure of a System